On a Special Type Nearly Quasi-Einstein Manifold

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Abstract: In the present paper, we consider a special type of nearly quasi-Einstein manifold denoted by $N(QE)_n$. Most of the sections are based on some properties of $N(QE)_n$. We give some theorems about these manifolds. In the last section, a special type nearly quasi-Einstein spacetime is investigated.

Keywords: Quasi-Einstein manifold, nearly quasi-Einstein manifold, spacetime.

1. Introduction

A non-flat $n$-dimensional Riemannian or a semi-Riemannian manifold $(M, g)$ ($n > 2$) is said to be an Einstein manifold if the condition

$$S(X, Y) = \frac{r}{n} g(X, Y)$$

holds on $M$, where $S$ and $r$ denote the Ricci tensor and the scalar curvature of $(M, g)$, respectively.

Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. For this reason, these manifolds have been studied by many authors.

A non-flat $n$-dimensional Riemannian manifold $(M, g)$ ($n > 2$) is defined to be a quasi-Einstein manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the following condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y)$$

where $a, b \in \mathbb{R}$ and $A$ is a non-zero 1-form such that

$$g(X, U) = A(X)$$

for all vector fields $X$ on $M$, [4]. Then $A$ is called the associated 1-form and $U$ is called the generator of the manifold.

Also M.C. Chaki and R.K. Maity [1] studied the quasi-Einstein manifolds by considering $a$ and $b$ as scalars such that $b \neq 0$ and $U$ as a unit vector field.

In 2008, U.C. De and A.K. Gazi [2] introduced the notion of nearly quasi-Einstein manifold. A non-flat $n$-dimensional Riemannian manifold $(M, g)$ ($n > 2$) is called a nearly quasi-Einstein manifold if its Ricci tensor $S$ of type $(0, 2)$ is not identically zero and satisfies the following condition
\[ S(X, Y) = a g(X, Y) + b E(X, Y) \]  
(1.4)

where \( a \) and \( b \) are non-zero scalars and \( E \) is a non-zero symmetric tensor of type (0,2).

Then \( E \) is called the associated tensor and \( a \) and \( b \) are called the associated scalars of \( M \). An \( n \)-dimensional nearly quasi-Einstein manifold is denoted by \( N(QE)_n \). An example of \( N(QE)_n \) has been given in [2].


In [8], R.N. Singh, M.K. Pandey, D. Gautam consider a type of nearly quasi-Einstein manifold whose associated tensor \( E \) of type (0,2) is in the form
\[ E(X, Y) = A(X)B(Y) + B(X)A(Y) \]  
(1.5)

where \( A \) and \( B \) are non-zero 1-forms associated with orthogonal unit vector fields \( V \) and \( U \), i.e.,
\[ g(U, U) = 1, \quad g(V, V) = 1 \quad \text{and} \quad g(U, V) = 0. \]  
(1.6)

These vector fields are defined by
\[ g(X, U) = A(X), \quad g(X, V) = B(X) \]
for every vector field \( X \).

In the present paper, we consider a special type of nearly quasi-Einstein manifold, \( N(QE)_n \), whose associated tensor \( E \) is of the form (1.5) with the condition (1.6). Some theorems about this manifold are proved and some properties are obtained.

2. A Special Type Nearly Quasi-Einstein Manifold

In this section, we consider a special type of \( N(QE)_n \) whose Ricci tensor satisfies the conditions (1.5) and (1.6), i.e., it satisfies the following condition
\[ S(X, Y) = a g(X, Y) + b[A(X)B(Y) + B(X)A(Y)] \]  
(2.1)

where \( A \) and \( B \) are non-zero 1-forms, \( a \) and \( b \) are non-zero scalars.

**Definition 1.** A vector field \( \xi \) in a Riemannian manifold \( M \) is called torse-forming if it satisfies the following condition
\[ \nabla_x \xi = \rho X + \phi(X)\xi \]  
(2.2)

where \( X \in TM \), \( \phi \) is a linear form and \( \rho \) is a function, [10].

In the local transcription, this reads
\[ \nabla_x \xi = \rho \delta^h_x + \xi^h \phi \]  
(2.3)

where \( \xi^h \) and \( \phi \) are the components of \( \xi \) and \( \phi \), and \( \delta^h_x \) is the Kronecker symbol.

A torse-forming vector field \( \xi \) is called
i) recurrent, if \( \rho = 0 \),
ii) concircular, if the form \( \phi \) is a gradient covector, i.e., there is a function \( \psi(x) \) such that \( \phi = d\psi(x) \),

iii) convergent, if it is concircular and \( \rho = \text{const.} \exp(\psi) \).

Therefore, recurrent vector fields are characterized by the following equation

\[
\nabla_X \xi = \phi(X) \xi.
\]

Also, from the Definition 1, for a concircular vector field \( \xi \), we get

\[
(\nabla_X \xi)X = \rho g(X,Y)
\]

for all \( X,Y \in TM \).

**Theorem 2.1.** Let \( V_n \) be a \( N(QE)_n \) satisfying the condition (2.1) and let \( U \) and \( V \) be the vector fields corresponding to the associated 1-forms \( A \) and \( B \), respectively. Thus, the vector fields \( U \) and \( V \) cannot be concircular vector fields.

**Proof.** We consider a special type \( N(QE)_n \) satisfying the condition (2.1). Let \( U \) and \( V \) corresponding to the associated 1-forms \( A \) and \( B \) be concircular vector fields, respectively. In local coordinates, thus we have

\[
\nabla_i A_j = \rho g_{ij}
\]

and

\[
\nabla_i B_j = \sigma g_{ij}
\]

where \( \rho \) and \( \sigma \) are non-zero scalar functions.

Taking the covariant derivative of the condition \( g(U,U) = 1 \), it is found that

\[
(\nabla_i A)A^i = 0
\]

where \( A^i = g^{ih} A_h \) and \( h \) is the arbitrary choice for indexing and the summation runs from 1 to \( n \).

Multiplying (2.6) by \( A^i \) and using the equation (2.8), we get

\[
\rho A = 0
\]

which contradicts to the fact that \( \rho \) is a non-zero scalar function and \( A \) is a non-zero 1-form. Similarly, it can be shown that the generator \( V \) cannot be a concircular vector field. In this case, \( N(QE)_n \) satisfying the condition (2.1) does not admit concircular vector fields \( U \) and \( V \) corresponding to the associated 1-forms \( A \) and \( B \), respectively. Hence, the proof is completed.

**Definition 2.** A quadratic conformal Killing tensor is defined as a second order symmetric tensor \( T \) satisfying the condition

\[
(\nabla_X T)(Y,Z) + (\nabla_Y T)(Z,X) + (\nabla_Z T)(X,Y) = \alpha(X)g(Y,Z) + \alpha(Y)g(Z,X) + \alpha(Z)g(X,Y)
\]

where \( \alpha \) is a 1-form, [9].

Now, we consider a \( N(QE)_n \), admitting a generator vector as a torse-forming vector field and the other be not. If we assume that the generator \( U \) is a torse-forming vector field, then we have from (1.6) and (2.3)
\[ \nabla_i A_j = \rho (g^i_j - A_i A_j) \]  \quad (2.10)

where \( \rho \) is a scalar function.

Taking the covariant derivative of the condition \( g(U, V) = 0 \) and using the equation (2.10), it can be seen that

\[ A'(\nabla_i B_i) = -\rho B_i. \] \quad (2.11)

By the aid of (2.9), (2.10) and (2.11), we prove the following theorem.

**Theorem 2.2.** Let \( V \) be a \( N(QE) \) satisfying the condition (2.1) and admitting the Ricci tensor as a quadratic conformal Killing tensor. If the vector field \( U \) generated by the 1-form \( A \) is a torse-forming vector field and the other vector field \( V \) generated by the 1-form \( B \) is not, then the vector field \( V \) is divergence-free.

**Proof.** Suppose that the Ricci tensor of a \( N(QE) \) satisfying the condition (2.1) is a quadratic conformal Killing tensor. In this case, in local coordinates, we have from (2.9)

\[ \nabla_i S_{ij} + \nabla_j S_{ij} + \nabla_j S_{ij} = \alpha_k g_{ij} + \alpha_i g_{jk} + \alpha_j g_{ki} \] \quad (2.12)

where \( \alpha \) is a 1-form.

Taking the covariant derivative of (2.1), we get

\[ \nabla_i S_{ij} = a_k g_{ij} + b_k (A_i B_j + A_j B_i) 
+ b_j (\nabla_i A_j B_j + A_i (\nabla_j B_j) + A_j (\nabla_i B_j) + A_j (\nabla_i B_j)) \] \quad (2.13)

where \( a \) and \( b \) are the associated scalars of this manifold and \( a_k = \partial_k a, b_k = \partial_k b \).

If the vector field \( U \) generated by the 1-form \( A \) is a torse-forming vector field, then we have the relation (2.10). Changing the indices by cyclic in (2.13), using (2.10) and (2.12), it can be obtained that

\[ \begin{align*}
(n + 2)(a_k + 2b \rho B_k - \alpha_k) g_{ij} + (a_i + 2b \rho B_i - \alpha_i) g_{jk} + (a_j + 2b \rho B_j - \alpha_j) g_{ki} \\
+ b_k (A_i B_j + A_j B_i) + b_j (A_k B_i + A_i B_j) + b_j (A_k B_i + A_i B_j) \\
+ b_j (A_i \nabla_j B_i) + A_i (\nabla_j B_i) + A_j (\nabla_i B_j) + A_j (\nabla_j B_j) + A_j (\nabla_i B_j) \\
- 2b \rho (A_i A_j B_k + A_j A_i B_k + A_i A_j B_k + A_j A_i B_k) = 0.
\end{align*} \] \quad (2.14)

Multiplying (2.14) by \( g^{ij} \) and considering (2.11), we get

\[ \begin{align*}
(n + 2)(a_k + 2b \rho B_k - \alpha_k) + 2b (A_i B_j + A_j B_i) \\
- 4b \rho B_k + 2b (A_i \nabla_j B_j + A_j \nabla_i B_j) + A_i \nabla_j B_j) = 0.
\end{align*} \] \quad (2.15)

Moreover, multiplying (2.15) by \( A^i \) and \( B^i \), respectively, and using the condition (1.6), we obtain the following equations

\[ \begin{align*}
(n + 2)(a_k - \alpha_k) A^i + 2b_k B^i + 2b \nabla_i B^i = 0
\end{align*} \] \quad (2.16)

\[ \begin{align*}
(n + 2)(a_k - \alpha_k) B^i + 2nb \rho + 2h_k A^i = 0.
\end{align*} \] \quad (2.17)

On the other hand, multiplying (2.14) by \( A'A'A^k \) and using (2.11), it is found that

\[ (a_k - \alpha_k) A^i = 0. \] \quad (2.18)
Multiplying (2.14) by $B^k A^k$, we find
\[(a_k - \alpha_k)A^k + 2b_k B^k = 0. \tag{2.19}\]
Since $b$ is a non-zero scalar function, from (2.16), (2.18) and (2.19), it can be seen that
\[\nabla_k B^k = 0.\]

Thus, the vector field $V$ generated by the 1-form $B$ is divergence-free. This completes the proof.

**Definition 3.** A non-flat $n$-dimensional Riemannian manifold $(M, g)$ ($n > 2$) is called a generalized Ricci-recurrent manifold if its Ricci tensor $S$ of type $(0,2)$ satisfies the condition
\[(\nabla_k S)(Y, Z) = \gamma(X) S(Y, Z) + \delta(X) g(Y, Z) \tag{2.20}\]
where $\gamma$ and $\delta$ are non-zero 1-forms, [3]. If $\delta = 0$, then the manifold reduces to a Ricci-recurrent manifold, [6].

**Theorem 2.3.** Let $N(QE)_n$ be a generalized Ricci-recurrent manifold. Thus, the vector fields $U$ and $V$ generated by the 1-forms $A$ and $B$ cannot be torse-forming vector fields.

**Proof.** We consider that $V_n$ is a $N(QE)_n$ satisfying the condition (2.1). In this case, in local coordinates, we have the equation (2.13) by Theorem 2.2. Let the vector field $U$ generated by the 1-form $A$ be a torse-forming vector field and the other be not. Then the relation (2.10) is satisfied. If we suppose that $V_n$ is a generalized Ricci-recurrent manifold, by the aid of (2.10), (2.13) and (2.20), we obtain
\[(a_k - \delta_k - a\gamma_k)g_{ij} + (b_k - b\gamma_k)(A_i B_j + A_j B_i) + b\rho(g_{ij} - A_i A_j)B_j + A_j (\nabla_k B_j) = 0 \tag{2.21}\]
where $\gamma_i$ and $\delta_i$ denote the components of the 1-forms $\gamma$ and $\delta$.

Multiplying (2.21) by $g^{ij}$ and using the condition (2.11), it can be seen that
\[a_k = \delta_k + a\gamma_k. \tag{2.22}\]
Moreover, multiplying (2.21) by $A' A'$ and using (1.6), we get
\[a_k - \delta_k - a\gamma_k + 2b A' (\nabla_k B_j) = 0. \tag{2.23}\]
By the aid of (2.11), (2.22) and (2.23), it is found that
\[b\rho B_k = 0\]
which contradicts to the fact that $b$ and $\rho$ are non-zero scalar functions and $B$ is a non-zero 1-form. Therefore, the vector field $U$ of this manifold cannot be a torse-forming vector field. By similar calculations it can be easily obtained that the vector field $V$ of this manifold also cannot be a torse-forming vector field. Thus, the proof is completed.

**3. A Special Type $N(QE)_n$ Spacetime**

In this section, we will examine $N(QE)_n$ spacetime which will be denoted by $N(QES)_n$ satisfying the condition (2.1).

The Einstein field equations (EFE) without cosmological constant is written as the following form
where $S$ is the Ricci tensor, $r$ is the scalar curvature, $g$ is the metric tensor, $k$ is a constant and $T$ is the energy-momentum tensor.

**Theorem 3.1.** In a $N(QES)_4$ satisfying the condition (2.1), the trace of the energy-momentum tensor is constant if and only if the associated scalar $a$ is constant.

**Proof.** Let us consider a $N(QES)_4$ satisfying the condition (2.1). From (3.1) and (2.1), it is obtained that

$$kT(X,Y) = S(X,Y) - \frac{r}{2} g(X,Y)$$

(3.1)

Moreover, using (2.1), the scalar curvature of a $N(QES)_4$ is found as

$$r = 4a.$$  

(3.3)

From (3.2) and (3.3), we have

$$kT(X,Y) = (a - \frac{r}{2}) g(X,Y) + b(A(X)B(Y) + A(Y)B(X)).$$

(3.2)

Contracting (3.4) over $X$ and $Y$, we obtain

$$\tilde{T} = -\frac{4}{k} a$$

(3.5)

where $\tilde{T}$ denotes the trace of the energy-momentum tensor.

It follows from (3.5) that if the associated scalar $a$ is constant, then the trace of the energy-momentum tensor is constant. The converse is also true. Hence, the proof is completed.

**Theorem 3.2.** In a perfect fluid $N(QES)_4$ spacetime satisfying the condition (2.1) with the constant associated scalar $a$, the change of the isotropic pressure is proportional to the change of the energy density.

**Proof.** In a perfect fluid spacetime, the energy-momentum tensor is in the form

$$T(X,Y) = (\sigma + p)\lambda(X)\lambda(Y) + pg(X,Y)$$

(3.6)

where $\sigma$ is the energy density, $p$ is the isotropic pressure and $\lambda$ is a non-zero 1-form such that $g(X,V) = \lambda(X)$ for all $X, V$ being the velocity vector field of the flow, that is, $g(V,V) = -1$. Also, $\sigma + p \neq 0$.

Using (3.6) in (3.1) and contracting the resulting equation over $X$ and $Y$, and considering the condition $g(V,V) = -1$ and (3.3), it can be seen that

$$3p - \sigma = -\frac{4}{k} a$$

(3.7)

where $a$ is the associated scalar of the manifold and $k$ is a constant.

If the associated scalar $a$ of $N(QES)_4$ is constant, then taking the covariant derivative of the equation (3.7) yields

$$3\nabla_p p = \nabla_p \sigma$$

(3.8)
for all vector fields $Z$.

Thus, the change of the isotropic pressure is proportional to the change of the energy density. This completes the proof.

References